# 2020 AIME II Problems & Solutions

## 2020 AIME II Problem 1

Find the number of ordered pairs of positive integers such that .

Solution 1

First, we find the prime factorization of , which is . The equation tells us that we want to select a perfect square factor of , . will be assigned by default. There are ways to select a perfect square factor of , thus our answer is .

Solution 2 (Official MAA)

Because , if , there must be nonnegative integers , , , and such that and . Then and The first equation has solutions corresponding to , and the second equation has solutions corresponding to . Therefore there are a total of ordered pairs such that .

## 2020 AIME II Problem 2

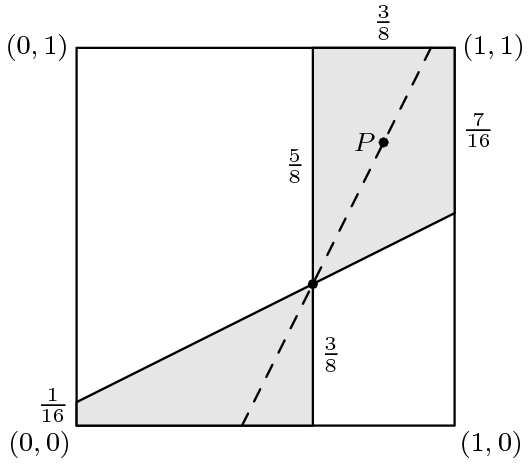
Let be a point chosen uniformly at random in the interior of the unit square with vertices at , and . The probability that the slope of the line determined by and the point is greater than or equal to can be written as , where and are relatively prime positive integers. Find .

Solution 1

The areas bounded by the unit square and alternately bounded by the lines through that are vertical or have a slope of show where can be placed to satisfy the condition. One of the areas is a trapezoid with bases and and height . The other area is a trapezoid with bases and and height . Then, .

Solution 2 (Official MAA)

The line through the fixed point with slope has equation . The slope between and the fixed point exceeds if falls within the shaded region in the diagram below consisting of two trapezoids with area Because the entire square has area the required probability is . The requested sum is .



## 2020 AIME II Problem 3

The value of that satisfies can be written as , where and are relatively prime positive integers. Find .

Solution 1

Let . Based on the equation, we get and . Expanding the second equation, we get . Substituting the first equation in, we get , so . Taking the 100th root, we get . Therefore, , and using the our first equation(), we get and the answer is . ~rayfish

Easiest Solution

Recall the identity (which is easily proven using exponents or change of base). Then this problem turns into Divide from both sides. And we are left with .Solving this simple equation we get .

Solution 2

Because we have that or Since and thus resulting in or We remove the base 3 logarithm and the power of 2 to yield or

Our answer is

Solution 3 (Official MAA)

Using the Change of Base Formula to convert the logarithms in the given equation to base yields Canceling the logarithm factors then yields which has solution The requested sum is .

## 2020 AIME II Problem 4

Triangles and lie in the coordinate plane with vertices , , , , , . A rotation of degrees clockwise around the point where , will transform to . Find .

Solution 1

After sketching, it is clear a rotation is done about . Looking between and , and . Solving gives . Thus .

Solution 2 (Official MAA)

Because the rotation sends the vertical segment to the horizontal segment , the angle of rotation is degrees clockwise. For any point not at the origin, the line segments from to and from to are perpendicular and are the same length. Thus a clockwise rotation around the point sends the point to the point . This has the solution . The requested sum is .

Solution 3

A degree rotation is obvious. Let’s look at and . They are very close to each other. Let’s join and with a line. Then construct a perpendicular bisector to with the midpoint being which is at . We also draw a point on the perpendicular bisector such that is . That point is the same distance to as is to but it is on a line perpendicular to Therefore is at . The sum is .

Solution 4

For the above reasons, the transformation is simply a rotation. Proceed with complex numbers on the points and . Let be the origin. Thus, and . The transformation from to is a multiplication of , which yields . Equating the real and complex terms results in the equations and . Solving,

Solution 5

We know that the rotation point has to be equidistant from both and so it has to lie on the line that is on the midpoint of the segment and also the line has to be perpendicular to . Solving, we get the line is . Doing the same for and , we get that . Since the point of rotation must lie on both of these lines, we set them equal, solve and get: ,. We can also easily see that the degree of rotation is since is initially vertical, and now it is horizontal. Also, we can just sketch this on a coordinate plane and easily realize the same. Hence, the answer is

## 2020 AIME II Problem 5

For each positive integer , let be the sum of the digits in the base-four representation of and let be the sum of the digits in the base-eight representation of . For example, , and . Let be the least value of such that the base-sixteen representation of cannot be expressed using only the digits through . Find the remainder when is divided by .

Solution 1

Let’s work backwards. The minimum base-sixteen representation of that cannot be expressed using only the digits through is , which is equal to in base 10. Thus, the sum of the digits of the base-eight representation of the sum of the digits of is . The minimum value for which this is achieved is . We have that . Thus, the sum of the digits of the base-four representation of is . The minimum value for which this is achieved is . We just need this value in base 10 modulo 1000. We get . Taking this value modulo , we get the final answer of . (If you are having trouble with this step, note that ) ~ TopNotchMath

Solution 2 (Official MAA)

First note that if is the least positive integer whose digit sum, in some fixed base , is , then is a strictly increasing function. This together with the fact that shows that is the least positive integer whose base-eight digit sum is 10. Thus , and is the least positive integer whose base-four digit sum is Therefore

## 2020 AIME II Problem 6

Define a sequence recursively by , , and for all . Then can be written as , where and are relatively prime positive integers. Find .

Solution 1

Let . Then, we have where and . By substitution, we find , , , , and . So has a period of . Thus . So, . ~mn28407

Solution 2 (Official MAA)

More generally, let the first two terms be and and replace and in the recursive formula by and , respectively. Then some algebraic calculation shows that so the sequence is periodic with period . Therefore The requested sum is .

Solution 3

Let us examine the first few terms of this sequence and see if we can find a pattern. We are obviously given and , so now we are able to determine the numerical value of using this information:

.

Now using this information, as well as the previous information, we are able to determine the value of :

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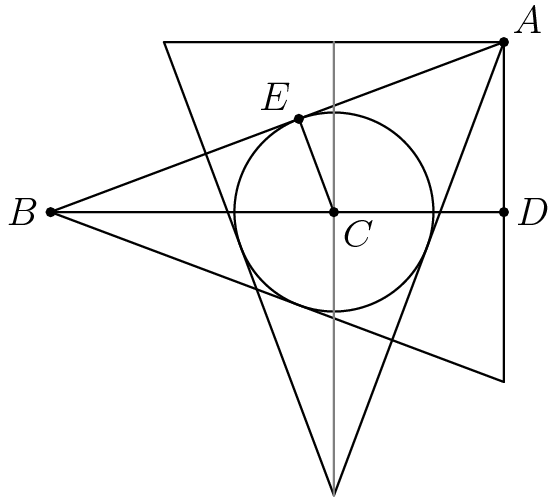
Alas, we have figured this sequence is period 5! Thus, let us take , which is , and therefore . According to the original problem statement, our answer is essentially .

## 2020 AIME II Problem 7

Two congruent right circular cones each with base radius and height have the axes of symmetry that intersect at right angles at a point in the interior of the cones a distance from the base of each cone. A sphere with radius lies withing both cones. The maximum possible value of is , where and are relatively prime positive integers. Find .

Solution (Official MAA)

Consider the cross section of the cones and sphere by a plane that contains the two axes of symmetry of the cones as shown below. The sphere with maximum radius will be tangent to the sides of each of the cones. The center of that sphere must be on the axis of symmetry of each of the cones and thus must be at the intersection of their axes of symmetry. Let be the point in the cross section where the bases of the cones meet, and let be the center of the sphere. Let the axis of symmetry of one of the cones extend from its vertex, , to the center of its base, . Let the sphere be tangent to at . The right triangles and are similar, implying that the radius of the sphere is The requested sum is .



## 2020 AIME II Problem 8

Define a sequence recursively by and for integers . Find the least value of such that the sum of the zeros of exceeds .

Solution (Official MAA)

First it will be shown by induction that the zeros of are the integers , where

This is certainly true for . Suppose that it is true for , and note that the zeros of are the solutions of , where is a nonnegative zero of . Because the zeros of form an arithmetic sequence with common difference so do the zeros of . The greatest zero of is so the greatest zero of is and the least is .

It follows that the number of zeros of is , and their average value is . The sum of the zeros of is Let , so the sum of the zeros exceeds if and only if Because is increasing for , the values and show that the requested value of is

## 2020 AIME II Problem 9

While watching a show, Ayako, Billy, Carlos, Dahlia, Ehuang, and Frank sat in that order in a row of six chairs. During the break, they went to the kitchen for a snack. When they came back, they sat on those six chairs in such a way that if two of them sat next to each other before the break, then they did not sit next to each other after the break. Find the number of possible seating orders they could have chosen after the break.

Solution 1 (Bash)

There are intersections that we must consider if we are to perform a PIE bash on this problem. Since we don’t really want to think that hard, and bashing does not take that long for this problem, we can write down half of all permutations that satisfy the conditions presented in the problem in “lexicographically next” order to keep track easily. We do this for all cases such that the first “person” is , and multiply by two, since the number of working permutations with as the first person is the same as if it were , hence, after doing such a bash, we get permutations that result in no consecutive letters being adjacent to each other. ~afatperson

Solution 2 (Official MAA)

Ayako (), Billy , Carlos , Dahlia , Ehuang , and Frank originally sat in the order . Let denote the set of seatings where and sit next to each other after the break. Then the required number of seating orders is given by the Inclusion-Exclusion Principle as Each term can be calculated separately.

1. Because there are terms, the sum is .
2. For , if , then must sit consecutively, so . There are terms that satisfy , so the sum is . If and are pairwise disjoint, then . There are terms, so the sum is .
3. If there are at least three pairs that sit next to each other, consider these three subcases: If the three pairs are consecutive, the sum is . If exactly two of the pairs are consecutive, the sum is . If none of the three pairs is consecutive, the sum is (d) If there are at least four pairs that sit next to each other, then if the pairs are consecutive, the sum is . If the pairs are not consecutive, then the sum is .
4. If all five pairs sit next to each other, the number is .

Therefore the required number of seating orders is

## 2020 AIME II Problem 10

Find the sum of all positive integers such that when is divided by , the remainder is .

Solution 1 (If you don’t remember the formula for sum of cubes)

We first note that since the remainder is and we are dividing by , must be greater than , meaning that has to be at least .

We then notice that we can pair the term with the term to factor it into using the sum of cubes formula, which is divisible by . We can do the same for the term with the term, the term with the , and so on, which are all divisible by . However, when is odd, we will have a middle term that is not paired with any other terms, which is not necessarily divisible by . Thus, we have two cases:

is even If is even, all terms that are greater than pair, as there are an even number of terms that are greater than . Therefore, all we need in order for the entire sequence to have a remainder of when divided by is to have a remainder of when divided by .

Evaluating as , all we need to be true is or that Thus, will be divisible by where . As is prime, must be equal to either or . If , we have that , which is not greater than or equal to , so that solution is extraneous.

If , we have that , which is , so one of our solutions is , and we are done with our first case.

is odd If is odd, the only term that does not pair is the arithmetic mean of the numbers under the cube of the largest and smallest terms that would pair, or . Therefore, as all other terms that are pair, the requirement that we have is Calculating and simplifying, we have that Now, we multiply both sides by . However, since multiplication is not reversible in modular arithmetic, we need to check whether any solutions are extraneous after solving. The congruence that we now have is As we know that is divisible by , what we need now is We now check each solution to see whether it works.

If , would be less than , so none of these solutions work. If , would be even, so that solution does not work for this case. Therefore, the only three solutions we need to check for this case are when , , or . We plug these values into the congruence before we multiplied both sides by to check.

If , we would need Calculating and factoring out , we have that As the right parenthesis is odd and , we know that this solution works, so we have another solution: .

If , we would need As the left hand side is odd, but all multiples of is even, this solution is therefore extraneous.

If , we would need Again, the left hand side is odd, and all multiples of are even, so this solution is extraneous.

Therefore, our final answer is .

Solution 2 (w/ formula)

Let . Then we have So, . Testing, the cases, only fails. This leaves .

Solution 3 (Official MAA 1)

The sum of the cubes from 1 to is For this to be equal to for some integer , it must be that so . But Thus is congruent to both and which implies that divides . Because , the only choices for are and Checking all three cases verifies that and work, but does not. The requested sum is .

Solution 4 (Official MAA 2)

The sum of the cubes of the integers from through is which, when divided by , has quotient with remainder If is not congruent to , then is an integer, and so divides , and . If , then is half of an integer, and letting for some integer gives Thus divides . It follows that , and . The requested sum is .

Solution 5

Using the formula for , Since divided by has a remainder of , Using the rules of modular arithmetic, Expanding the left hand side, This means that is divisible by .

Dividing polynomials,

Note that and (because the remainder when dividing by is , so must be greater than ), so all options can be eliminated. Checking all 3 cases, and work; fails.

Therefore, the answer is .

## 2020 AIME II Problem 11

Let , and let and be two quadratic polynomials also with the coefficient of equal to . David computes each of the three sums , , and and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If , then , where and are relatively prime positive integers. Find .

Solution 1

Let and . We can write the following: Let the common root of be ; be ; and be . We then have that the roots of are , the roots of are , and the roots of are .

By Vieta’s, we have:

Subtracting from , we get . Adding this to , we get . This gives us that from . Substituting these values into and , we get and . Equating these values, we get . Thus, our answer is . ~ TopNotchMath

Solution 2

We know that .

Since , the constant term in is . Let .

Finally, let .

. Let its roots be and .

Let its roots be and .

. Let its roots be and .

By vietas,

We could work out the system of equations, but it’s pretty easy to see that .

Solution 3 (Official MAA)

Let the common root of and be , the common root of and be , and the common root of and be . Because and are both roots of and has leading coefficient , it follows that Similarly, and . Adding these three equations together and dividing by yields so Similarly, Comparing the coefficients yields , and comparing the constant coefficients yields . The fact that implies that . Adding these two equations yields , and so substituting back in to solve for gives . Finally, The requested sum is . Note that and .

## 2020 AIME II Problem 12

Let and be odd integers greater than An rectangle is made up of unit squares where the squares in the top row are numbered left to right with the integers through , those in the second row are numbered left to right with the integers through , and so on. Square is in the top row, and square is in the bottom row. Find the number of ordered pairs of odd integers greater than with the property that, in the rectangle, the line through the centers of squares and intersects the interior of square .

Solution 1

Let us take some cases. Since and are odds, and is in the top row and in the bottom, has to be , , , or . Also, taking a look at the diagram, the slope of the line connecting those centers has to have an absolute value of . Therefore, .

If , can range from to . However, divides , so looking at mods, we can easily eliminate and . Now, counting these odd integers, we get .

Similarly, let . Then can range from to . However, , so one can remove and . Counting odd integers, we get .

Take . Then, can range from to . However, , so one can verify and eliminate and . Counting odd integers, we get .

Let . Then can vary from to . However, . Checking that value and the values around it, we can eliminate . Counting odd integers, we get .

Add all of our cases to get

Solution 2 (Official MAA)

Because square is in the bottom row, it follows that . Moreover, because square is in the top row, and square is not in the top row, . In particular, because the number of rows in the rectangle must be odd, must be one of or

For each possible choice of and , let denote the line through the centers of squares and Note that for odd values of , the line passes through the center of square Thus intersects the interior of cell exactly when its slope is strictly between and . The line is vertical whenever square is the th square in the bottom row of the rectangle. This would happen for when , respectively. When is 1 greater than or 1 less than these numbers, the slope of is or , respectively. In all other cases the slope is strictly between and The admissible values for for each possible value of are given in the following table.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| m | minimum n | maximum n | avoided n | number of odd n |
| 3 | 667 | 999 | 899 | 900 |
| 5 | 400 | 499 | 449 | 450 |

\ 7 & 286 & 333 &$ (Error compiling LaTeX. ! Misplaced alignment tab character &.)299, 300, 301$& 22\\ 9 & 223 & 249 &$ (Error compiling LaTeX. ! Misplaced alignment tab character &.)224, 225, 226$& 13\\ \hline \end{tabular}$ (Error compiling LaTeX. ! Misplaced alignment tab character &.) This accounts for rectangles.

## 2020 AIME II Problem 13

Convex pentagon has side lengths , , and . Moreover, the pentagon has an inscribed circle (a circle tangent to each side of the pentagon). Find the area of .

Solution 1 (Misplaced problem?)

Assume the incircle touches , , , , at respectively. Then let , , . So we have , and =7, solve it we have , , . Let the center of the incircle be , by SAS we can proof triangle is congruent to triangle , and triangle is congruent to triangle . Then we have , . Extend , cross ray at , ray at , then by AAS we have triangle is congruent to triangle . Thus . Let , then . So by law of cosine in triangle and triangle we can obtain , solved it gives us , which yield triangle to be a triangle with side length 15, 15, 24, draw a height from to divides it into two triangles with side lengths 9, 12, 15, so the area of triangle is 108. Triangle is a triangle with side lengths 6, 8, 10, so the area of two of them is 48, so the area of pentagon is .

Solution 2 (Complex Bash)

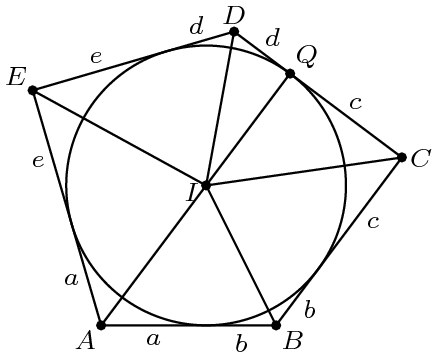
Suppose that the circle intersects , , , , and at , , , , and respectively. Then , , , , and . So , , , , and . Then , so . Then we can solve for each individually. , , , , and . To find the radius, we notice that , or . Each of these angles in this could be represented by complex numbers. When two complex numbers are multiplied, their angles add up to create the angle of the resulting complex number. Thus, is real. Expanding, we get: , then . On the last expanding, we only multiply the reals with the imaginaries and vice versa, because we only care that the imaginary component equals 0. . . must equal 4, as r cannot be negative or be approximately equal to 1. Thus, the area of is

Solution 3 (Guess)

This pentagon is very close to a regular pentagon with side lengths . The area of a regular pentagon with side lengths is . is slightly greater than given that is slightly less than . is then slightly greater than . We will approximate that to be . The area is now roughly , but because the actual pentagon is not regular, but has the same perimeter of the regular one that we are comparing to we can say that this is an overestimate on the area and turn the into thus turning the area into which is and since is a multiple of the semiperimeter , we can safely say that the answer is most likely .

Solution 4 (Official MAA 1)

Let be the inscribed circle, be its center, and be its radius. The area of is equal to its semiperimeter, times , so the problem is reduced to finding . Let be the length of the tangent segment from to , and analogously define , , , and . Then , , and , with a total of . Hence , , and . It follows that and . Let be the point where is tangent to . Then . The sum of the internal angles in polygons and are equal, so , which implies that must be . Therefore points , , and are collinear.



Because , it follows that Another expression for can be found as follows. Note that and , so and Applying the Law of Cosines to and gives and Hence

yielding equivalently Substituting gives the quadratic equation , with solutions , and . The solution corresponds to a five-pointed star, which is not convex. Indeed, if , then , , and are less than implying that , , and are acute, which cannot happen in a convex pentagon. Thus and . The requested area is .

Solution 5 (Official MAA 2)

Define , , , , , and as in Solution 4. Then, as in Solution 4, , , , , and . Let , , and . It follows that , so . Thus , , and . By the Tangent Addition Formula, and Therefore which simplifies to . Then the solution proceeds as in Solution 4.

Solution 6 (Official MAA 3)

Define , , , , , and as in Solution 4. Note that Hence Therefore Simplifying this equation gives the same quadratic equation in as in Solution 4.

## 2020 AIME II Problem 14

For a real number let be the greatest integer less than or equal to , and define to be the fractional part of . For example, and . Define , and let be the number of real-valued solutions to the equation for . Find the remainder when is divided by .

Solution 1

It’s not too hard to see that, is One can see an easy combinatorical argument exists which is the official solution, but I will present another solution here for the sake of variety.

Applying algebraic manipulation and the hockey-stick identity times gives

Hence, $N = \frac{2005 \cdot 2004 \cdot 2003}{3 \cdot 2\cdot 1} \equiv 10 (\textbf{mod} \hskip .2cm 1000)$

Solution 2

To solve , we need to solve where , and to solve that we need to solve where .

It is clear to see for some integer there is exactly one value of in the interval where . To understand this, imagine the graph of on the interval The graph starts at , is continuous and increasing, and approaches . So as long as , there will be a solution for in the interval.

Using this logic, we can find the number of solutions to . For every interval where there will be one solution for in that interval. However, the question states , but because doesn’t work we can change it to . Therefore, , and there are solutions to .

We can solve similarly. to satisfy the bounds of , so there are solutions to , and to satisfy the bounds of .

Going back to , there is a single solution for z in the interval , where . (We now have an upper bound for because we know .) There are solutions for , and the floors of these solutions create the sequence

Lets first look at the solution of where . Then would have solutions, and the floors of these solutions would also create the sequence .

If we used the solution of where , there would be solutions for . If we used the solution of where , there would be solutions for , and so on. So for the solution of where , there will be solutions for

If we now look at the solution of where , there would be solutions for . If we looked at the solution of where , there would be solutions for , and so on.

The total number of solutions to is . Using the hockey stick theorem, we see this equals , and when we take the remainder of that number when divided by , we get the answer,

Solution 3 (Official MAA)

For any nonnegative integer , the function increases on the interval , with and for every in this interval. On this interval , which takes on every real value in the interval exactly once. Thus for each nonnegative real number , the equation has exactly one solution for every .

For each integer there is exactly one with such that ; likewise for each integer there is exactly one with and such that . Finally, for each integer there is exactly one with , , and such that .

Thus has exactly one solution with for each triple of integers with , noting that is not a solution. This nondecreasing ordered triple can be identified with a multiset of three elements of the set of integers , which can be selected in ways. Thus $N=\frac{2005\cdot 2004\cdot 2003}{6}\equiv 10\hskip -.2cm \mod {1000}.$

## 2020 AIME II Problem 15

Let be an acute scalene triangle with circumcircle . The tangents to at and intersect at . Let and be the projections of onto lines and , respectively. Suppose , , and . Find .

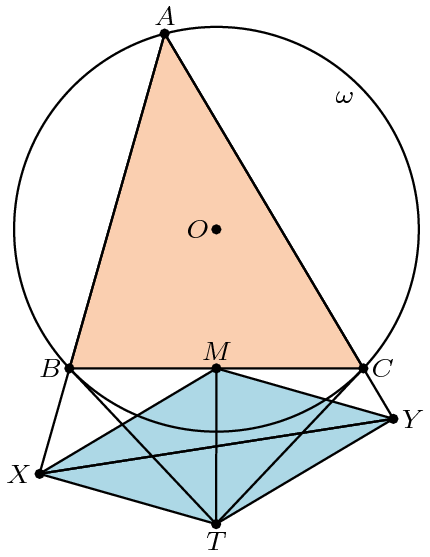
Solution 1

Assume to be the center of triangle , cross at , link , . Let be the middle point of and be the middle point of , so we have . Since , we have . Notice that , so , and this gives us . Since is perpendicular to , and cocycle (respectively), so and . So , so , which yields So same we have . Apply Ptolemy theorem in we have , and use Pythagoras theorem we have . Same in and triangle we have and . Solve this for and and submit into the equation about , we can obtain the result .

(Notice that, is a parallelogram, which is an important theorem in Olympiad, and there are some other ways of computation under this observation.)

Solution 2 (Official MAA)

Let denote the midpoint of . The critical claim is that is the orthocenter of , which has the circle with diameter as its circumcircle. To see this, note that because , the quadrilateral is cyclic, it follows that implying that . Similarly, . In particular, is a parallelogram.



Hence, by the Parallelogram Law, But . Therefore